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INVARIANTS OF A GENERAL BRANCHED COVER OF $\mathbf{P}^1$

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Abstract. We investigate the resolution of a *general* branched cover $\alpha\colon C \to \mathbf{P}^1$ in its relative canonical embedding $C \subset \mathbf{P}E$. We conjecture that the syzygy bundles appearing in the resolution are balanced for a general cover, provided that the genus is sufficiently large compared to the degree. We prove this for the *Casnati-Ekedahl* bundle, or *bundle of quadrics* F—the first bundle appearing in the resolution of the ideal of the relative canonical embedding. Furthermore, we prove the conjecture for all syzygy bundles in the resolution when the genus satisfies $g = 1 \mod d$.

1. Introduction

Every degree d branched cover $\alpha\colon C \to \mathbf{P}^1$ from a genus g curve C has a natural *relative canonical embedding*

$$\iota\colon C \hookrightarrow \mathbf{P}E$$

into a projective bundle $\mathbf{P}E$ over $\mathbf{P}^1$, where E is the classical *Tschirnhausen bundle* - the dual of the sheaf of traceless functions on C, viewed as an $\mathcal{O}_{\mathbf{P}^1}$-module. Geometrically, the scroll $\mathbf{P}E$ is swept out by the spans

$$\langle \alpha^{-1}(t) \rangle \subset \mathbf{P}^{g-1}$$

for all $t \in \mathbf{P}^1$. The embedding ι is such that the projection $\pi\colon \mathbf{P}E \to \mathbf{P}^1$ induces the map α on C, as illustrated in fig. 1.

The ideal sheaf $\mathcal{I}_C$ of $C \subset \mathbf{P}E$ has a well known relative minimal resolution, due to a theorem of Schreyer [Sch86], generalized later by Casnati and Ekedahl [CE96]. This resolution of $C \subset \mathbf{P}E$ involves *syzygy bundles* N_i, $i = 1, \ldots, d-2$, which are vector bundles on the target $\mathbf{P}^1$. The splitting types of N_i are then fundamental algebro-geometric invariants naturally attached to any branched cover $\alpha\colon C \to \mathbf{P}^1$.

Since the set of all branched covers of fixed degree and genus forms the irreducible *Hurwitz space* $\mathcal{H}_{d,g}$, it follows that the syzygy bundles N_i have a generic isomorphism type for a Zariski open subset of covers. Our general objective is to understand the generic splitting types of the syzygy bundles.

It is reasonable to expect that the bundles N_i are *balanced* for a general cover, in the sense that the largest degree of a summand of N_i is at most one

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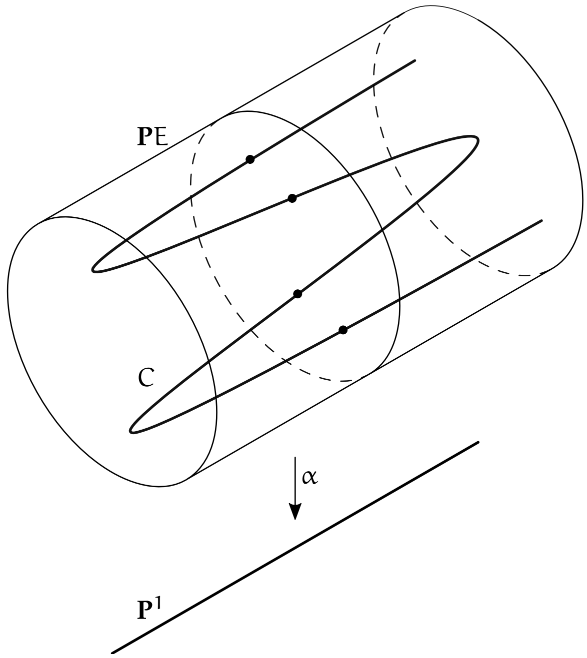

FIGURE 1. A cover sitting in its relative canonical scroll.

more than the smallest degree. This is succinctly written as

$$h^1(\mathrm{End}\, N_i) = 0.$$

Being balanced is certainly an open condition, in the sense that in a "typical" family of vector bundles on $\mathbf{P}^1$, one expects an open set of bundles to be balanced. With this expectation in mind, and with the aid of explicit examples in low genera and degree, we make the following conjecture:

Conjecture A. *The general cover* $[\alpha\colon C \to \mathbf{P}^1] \in \mathcal{H}_{d,g}$ *has balanced syzygy bundles when* $g \gg d$.

Remark 1.1. Before considering the syzygy bundles N_i, one might ask about the generic behavior of the Tschirnhausen bundle E. A theorem of Coppens [Cop99] and Ballico [Bal89] shows that E is balanced for a generic cover.

Remark 1.2. Conjecture A bears a striking resemblence to Green's conjecture on syzygies of canonical curves. It seems difficult to precisely identify the precise relationship between the two conjectures. [Bop14a].

This paper focuses specifically on the first bundle N_1, which we refer to simply as F. F is known as the *bundle of quadrics* or the *Casnati-Ekedahl bundle* - for simplicity we will often call it the F-bundle. The geometric interpretation of F is very simple: F parametrizes the vector space of quadrics containing the d points of C in the $\mathbf{P}^{d-2}$ fibers of $\mathbf{P}E$.

In recent work of Bopp and Hahn [Bop14b], the precise necessary relationship between g and d is studied within the regime where the Brill-Noether number satisfies $\rho(g, d, 1) \geq 0$ and $g > d + 1$.

In this (g, d)-regime, they conclude that the generic bundle of quadrics N_1 is balanced if and only if $\rho > 0$ and $(k - \rho - \frac{7}{2})^2 - 2d + \frac{23}{4} > 0$.

Furthermore, with the use of their *Macaulay2*-Package [BH15], their experimentation leads them to make the following refinement of conjecture A:

Conjecture B. *(Bopp, Hahn* [Bop14b]*)*

(1) If $\rho \leq 0$ then the bundle N_1 is balanced for a general cover $[\alpha\colon C \to \mathbf{P}^1] \in \mathcal{H}_{d,g}$.

(2) If $\rho \geq 0$, and if we write $N_i = \oplus \mathcal{O}_{\mathbf{P}^1}(a_j^{(i)})$, then the following bound holds for a general $\alpha \in \mathcal{H}_{d,g}$:

$$\max_{j,l} |a_j^{(i)} - a_l^{(i)}| \leq \min\{g - d - 1, i + 2\}.$$

Remark 1.3. Notice that the $g = d + 1$ case of part (2) is an instance of proposition 2.4.

Our main result is the proof of the N_1 case of Conjecture A:

Main Theorem. *Let $g \geq (d-3)(d-1)$. Then the bundle of quadrics F of a general cover $\alpha \in \mathcal{H}_{d,g}$ is balanced.*

The proof proceeds via degeneration. We begin in the first section by reviewing the Casnati-Ekedahl structure theorem, and then studying the bundle of quadrics for covers of genus 0 and 1. We then glue such covers together in a "chain" to create an admissible cover $\alpha\colon X \to P$ where P is a chain of $\mathbf{P}^1$'s.

Once we have constructed this cover, we must argue that its F-bundle, a vector bundle on the chain of rational curves P, is balanced. In order to do this, we first provide a general criterion for a vector bundle V on a chain P to satisfy $h^1(\operatorname{End} V) = 0$. Section 3 is devoted to establishing this criterion.

We are then left with the task of verifying that these conditions are met for the F-bundle of $\alpha\colon X \to P$. This, in turn, leads us to considering a degree 2 maximal rank problem for "maximally connected chains" of rational normal curves in projective space. Section 4 is devoted to establishing the connection between our original problem of showing $h^1(\operatorname{End} F) = 0$ with the degree 2 maximal rank problem. We settle the aforementioned maximal rank problem in section 5.

Throughout, we will work over an algebraically closed field k of characteristic zero.

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2. Preliminaries

We review the basic Casnati-Ekedahl structure theorem of branched covers of algebraic varieties. We then examine the Casnati-Ekedahl resolution of covers $\alpha\colon C \to \mathbf{P}^1$ when the genus $g(C)$ is 0 or 1.

2.1. **The Casnati-Ekedahl Resolution.** Let X and Y be integral schemes and $\alpha\colon X \to Y$ a finite flat Gorenstein morphism of degree $d \geq 3$. The map α gives an exact sequence

$$0 \to \mathcal{O}_Y \to \alpha_*\mathcal{O}_X \to {E_\alpha}^\vee \to 0, \tag{1}$$

where $E = E_\alpha$ is a vector bundle of rank $(d-1)$ on Y, called the *Tschirnhausen bundle* of α. Denote by ω_α the dualizing sheaf of α. Applying $\mathrm{Hom}_Y(-, \mathcal{O}_Y)$ to (1), we get

$$0 \to E \to \alpha_*\omega_\alpha \to \mathcal{O}_Y \to 0. \tag{2}$$

The map $E \to \alpha_*\omega_\alpha$ induces a map $\alpha^*E \to \omega_\alpha$.

Theorem 2.1. [CE96, Theorem 2.1] *In the above setup, $\alpha^*E \to \omega_\alpha$ gives an embedding $\iota\colon X \to \mathbf{P}E$ with $\alpha = \pi \circ \iota$, where $\pi\colon \mathbf{P}E \to Y$ is the projection. Moreover, the subscheme $X \subset \mathbf{P}E$ can be described as follows.*

(a) The resolution of $\mathcal{O}_X$ as an $\mathcal{O}_{\mathbf{P}E}$-module has the form

$$\begin{aligned} 0 \to \pi^*N_{d-2}(-d) \to \pi^*N_{d-3}(-d+2) \to \pi^*N_{d-4}(-d+3) \to \dots \\ \dots \to \pi^*N_2(-3) \to \pi^*N_1(-2) \to \mathcal{O}_{\mathbf{P}E} \to \mathcal{O}_X \to 0, \end{aligned} \tag{3}$$

where the N_i are vector bundles on Y. Restricted to a point $y \in Y$, this sequence is the minimal free resolution of length d zero dimensional scheme $X_y \subset \mathbf{P}E_y \cong \mathbf{P}^{d-2}$.

(b) The ranks of the N_i are given by

$$\operatorname{rk} N_i = \frac{i(d-2-i)}{d-1}\binom{d}{i+1},$$

(c) We have $N_{d-2} \cong \det E$.

(d) Furthermore, the resolution is symmetric, that is, isomorphic to the resolution obtained by applying the functor $\mathrm{Hom}_{\mathcal{O}_{\mathbf{P}E}}(-, \pi^*N_{d-2}(-d))$.

Remark 2.2. We will only use the case $Y = \mathbf{P}^1$, which was discovered by Schreyer in [Sch86, Corollary 4.4].

The branch divisor of $\alpha\colon X \to Y$ is given by a section of $(\det E)^{\otimes 2}$. In particular, if X is a curve of (arithmetic) genus g, α has degree d, and $Y = \mathbf{P}^1$, then

$$\operatorname{rk} E = d-1 \text{ and } \deg E = g+d-1.$$

We will be concerned with the vector bundle N_1, which from here onwards we denote by F, and refer to as the *bundle of quadrics*.

Notice that, by twisting the Casnati-Ekedahl resolution (3) by $\mathcal{O}_{\mathbf{P}E}(2)$ and then applying π_*, we obtain an exact sequence

$$0 \to F \to \operatorname{Sym}^2 E \to \alpha_*(\omega_\alpha^{\otimes 2}) \to 0. \tag{4}$$

From this sequence, it is easy to see that $\operatorname{rk} F = d(d-3)/2$ and $\deg F = (d-3)(g+d-1)$.

Theorem 2.1 allows us to associate to every cover $[\alpha\colon C \to \mathbf{P}^1] \in \mathcal{H}_{d,g}$ a collection of vector bundles $(E, F, ..., N_{d-2})$ on $\mathbf{P}^1$. Conjecture A expresses the natural expectation that these vector bundles are all *balanced* for a general cover α (we say a vector bundle V is balanced if $h^1(\operatorname{End} V) = 0$).

To fix notation, we let ζ denote the divisor class associated to the line bundle $\mathcal{O}_{\mathbf{P}E}(1)$ on a projective bundle $\mathbf{P}E$. We consider $\mathbf{P}E$ to be the scheme $\operatorname{Proj}(\operatorname{Sym}^* E)$, as in [Har77]. We let f denote the class of a fiber of the projection $\pi\colon \mathbf{P}E \to \mathbf{P}^1$.

2.2. **Evidence.** We explain the rationale behind conjecture A.

First, when the degree is at most 5, it is known that F is generically balanced: [DP15, Bop14b]. In these cases, F is (up to twist) the only bundle appearing in the Casnati-Ekedahl resolution, due to the symmetry of the resolution.

When $d = 6$ and $g = 4$, the F-bundle is generically *not* balanced:

Example 2.3. Let $\alpha\colon C \to \mathbf{P}^1$ be a general degree 6 cover, with C a genus 4, non-hyperelliptic curve. Then $E = \mathcal{O}(1) \oplus \mathcal{O}(2)^{\oplus 4}$. Since F is a sub vector bundle of $\operatorname{Sym}^2 E$, we conclude that the degree of any summand of F cannot exceed 4.

On the other hand, since $\deg F = 27$ and $\operatorname{rk} F = 9$, the bundle F is imbalanced if and only if $\mathcal{O}(4)$ is a summand of F.

We show that $H^0(F(-4)) \neq 0$ by considering the sequence (4). We want to show that the map $H^0(\mathbf{P}^1, \operatorname{Sym}^2 E(-4)) \to H^0(\mathbf{P}^1, \alpha_*(\omega_\alpha)(-4))$ is not injective. But we can identify this map with:

$$H^0(\mathbf{P}E, 2\zeta - 4f) \to H^0(C, (2\zeta - 4f)|_C) \tag{5}$$

The linear system $|\zeta - 2f|$ on $\mathbf{P}E$ restricts to the full canonical series on C. Furthermore, the sections of $2\zeta - 4f$ are obtained by taking sums of products of sections of $|\zeta - 2f|$. Since the canonical model of C lies on a unique quadric in $\mathbf{P}^3$, we see that there is a unique element of $|2\zeta - 4f|$ containing C. This

means that the map (5) is not injective, which implies that F contains an $\mathcal{O}(4)$ summand. Therefore F is not balanced.

The above example suggests that when the genus is small compared to the degree, the F-bundle may be imbalanced. This example can be generalized as long as g is somewhat small compared to d, and it explains the inequality $g \gg d$ in conjecture A.

Substantial support for the conjecture is provided by the following proposition:

Proposition 2.4. *Conjecture A is true whenever* $g = 1 \mod d$*. In fact, for a general cover* $\alpha\colon C \to \mathbf{P}^1$*, the syzygy bundles* N_i *are* perfectly *balanced, i.e.*

$$N_i = \mathcal{O}_{\mathbf{P}^1}(k_i)^{\oplus r_i}$$

for some integers k_i, r_i*.*

We'll need the following basic lemma about elliptic normal curves of degree d in $\mathbf{P}^{d-1}$:

Lemma 2.5. *Let* $E \subset \mathbf{P}^{d-1}$ *be a degree* d *smooth elliptic normal curve, let* H *be a hyperplane in* $\mathbf{P}^{d-2}$*, and let* $Z := H \cap E$*.*

(a) If

$$0 \to F_1 \to \ldots \to F_k \to \mathcal{I}_{E\subset\mathbf{P}^{d-1}} \to 0 \tag{6}$$

is the minimal free resolution of $E \subset \mathbf{P}^{d-1}$*, then*

$$0 \to F_1|_H \to \ldots \to F_k|_H \to \mathcal{I}_{Z\subset H} \to 0$$

is the minimal free resolution of $Z \subset H$*.*

(b) The syzygy bundles F_i *appearing in the minimal free resolution* (6) *are of the form* $\mathcal{O}_{\mathbf{P}^{d-1}}(-m_i)^{\oplus r_i}$ *for some integers* m_i *and* r_i*, i.e. all summands of* F_i *have the same degree.*

Proof. There is an exact sequence

$$0 \to \mathcal{I}_{E\subset\mathbf{P}^{d-1}}(-H) \to \mathcal{I}_{E\subset\mathbf{P}^{d-1}} \to \mathcal{I}_{Z\subset H} \to 0.$$

Take the minimal resolution of $\mathcal{I}_{E\subset\mathbf{P}^{d-1}}$:

$$0 \to F_1 \to \ldots \to F_k \to \mathcal{I}_{E\subset\mathbf{P}^{d-1}} \to 0$$

And form the following diagram

$$\begin{tikzcd}
0 \arrow[r] & F_1(-H) \arrow[r] \arrow[d] & \cdots \arrow[r] & F_k(-H) \arrow[r] \arrow[d] & \mathcal{I}_{E\subset\mathbf{P}^{d-1}}(-H) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & F_1 \arrow[r] \arrow[d] & \cdots \arrow[r] & F_k \arrow[r] \arrow[d] & \mathcal{I}_{E\subset\mathbf{P}^{d-1}} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & F_1|_H \arrow[r] & \cdots \arrow[r] & F_k|_H \arrow[r] & \mathcal{I}_{Z\subset H} \arrow[r] & 0
\end{tikzcd}$$

The top two rows are exact, and so is every column. Therefore, the bottom row is exact as well, and hence a resolution for $\mathcal{I}_{Z\subset H}$. The bottom resolution is minimal because the degrees of the F_k strictly increase with k.

The second part of the lemma is a well-known computation of the minimal free resolution of an elliptic normal curve found, for example, in [Eis]. $\square$

Proof of proposition 2.4. We essentially apply lemma 2.5 relatively over $\mathbf{P}^1$. Let $C \subset E \times \mathbf{P}^1$ be a smooth bi-degree (d, m) curve; this means that C intersects the elliptic curve rulings d times. These intersections are all in the same linear system $|D|$ of degree d on E. By adjunction, the curve C is seen to have genus $g = d(m-1)+1$.

We can embed the surface $E \times \mathbf{P}^1$ into $\mathbf{P}^{d-1} \times \mathbf{P}^1$ where the first factor is embedded by the linear system $|D|$, i.e. $E \times \mathbf{P}^1$ is a constant family of degree d elliptic normal curves in $\mathbf{P}^{d-1} \times \mathbf{P}^1$.

Let $\pi\colon \mathbf{P}^{d-1} \times \mathbf{P}^1 \to \mathbf{P}^1$ denote the projection. Then the ideal sheaf $\mathcal{I}_{E\times\mathbf{P}^1}$ has a "relative" minimal resolution which is pulled back via the projection to $\mathbf{P}^{d-1}$:

$$0 \to F_1 \to \ldots \to F_k \to \mathcal{I}_{E\times\mathbf{P}^1\subset\mathbf{P}^{d-1}\times\mathbf{P}^1} \to 0 \tag{7}$$

Let h denote the pullback of the hyperplane class on $\mathbf{P}^{d-1}$ to the product $\mathbf{P}^{d-1}\times\mathbf{P}^1$, and let f denote the class of a fiber of π. Then the curve $C \subset E\times\mathbf{P}^1$ is the intersection of $E\times\mathbf{P}^1$ with some relative hyperplane divisor $\Lambda \subset \mathbf{P}^{d-1}\times\mathbf{P}^1$ with divisor class $h + mf$.

The curve C, viewed as a subscheme of the $\mathbf{P}^{d-2}$-bundle Λ, is a branched cover in its relative canonical embedding. This follows because, by lemma 2.5, the fibers $C\cap\mathbf{P}^{d-2} \subset \mathbf{P}^{d-2}$ are length d arithmetically Gorenstein subschemes of $\mathbf{P}^{d-2}$, a property which characterizes the relative canonical embedding [CE96].

We then restrict sequence (7) to the relative hyperplane Λ, and use lemma 2.5 in each fiber of π to conclude that

$$0 \to F_1|_\Lambda \to \ldots \to F_k|_\Lambda \to \mathcal{I}_{C\subset\Lambda} \to 0 \tag{8}$$

is the Casnati-Ekedahl resolution of $C \subset \Lambda$. If we write

$$F_i|_\Lambda = \pi^* N_i(-m_i)$$

we see that each bundle N_i is in fact *perfectly* balanced, i.e.

$$N_i = \mathcal{O}_{\mathbf{P}^1}(k_i)^{\oplus r_i}$$

for some integer k_i, where r_i is given in item (b) of Theorem 2.1.

$\square$

Remark 2.6. Proposition 2.4 and lemma 2.5 indicate a direct relationship between invariants of branched covers and properties of genus 1 fibrations.

Let $f\colon S \to \mathbf{P}^1$ be a generically smooth family of arithmetic genus 1 curves, and let $C \subset S$ be a smooth d-section of the map f, i.e. C is a smooth curve whose intersection number with the fibers of S is d.

Then the pair $(f\colon S \to \mathbf{P}^1, C)$ gives rise to the natural embedding

$$S \hookrightarrow \mathbf{P}(f_*\mathcal{O}_S(C)) \tag{9}$$

over $\mathbf{P}^1$. The projective bundle $\mathbf{P}(f_*\mathcal{O}_S(C))$ is a $\mathbf{P}^{d-1}$-bundle, and each fiber of f embeds as a possibly singular degree d elliptic normal curve. Furthermore, there exists a relative hyperplane $\Lambda \subset \mathbf{P}(f_*\mathcal{O}_S(C))$ whose intersection with S is precisely the curve C.

The main observation, following from lemma 2.5, is that the curve C, viewed as a branched cover of $\mathbf{P}^1$, is embedded via its relative canonical embedding in the $\mathbf{P}^{d-2}$-bundle Λ.

As an analogy, the reader should recall the relationship between the syzygies of a canonical curve $C \subset \mathbf{P}^{g-1}$ and a K3 surface $S \subset \mathbf{P}^g$ containing C. A generic version of Green's conjecture was proved by Voisin in [Voi02] using this relationship.

A dimension count says that a general branched cover $\alpha\colon C \to \mathbf{P}^1$ does not occur as a d-section in a genus 1 fibration. However, the proof of proposition 2.4 shows us that *if the syzygy bundles for* $S \subset \mathbf{P}(f_*\mathcal{O}_S(C))$ *are balanced, then the same follows for* $C \subset \Lambda$.

This suggests a more attractive approach for proving conjecture A: Show that the generic pair $(f\colon S \to \mathbf{P}^1, C)$ has balanced syzygies in its relative embedding (9). This is the subject of future work.

2.3. **Covers of genus 0 and 1.**

We will now gather some relevant facts about genus 0 and genus 1 covers of $\mathbf{P}^1$.

Lemma 2.7. *Let* $\alpha\colon R \to \mathbf{P}^1$ *and* $\beta\colon X \to \mathbf{P}^1$ *be degree* d *covers where* R *is a smooth rational curve and* X *is a smooth elliptic curve. Then:*

- *(a)* $E_\alpha = \mathcal{O}(1)^{\oplus d-1}$,
- *(b)* $F_\alpha = \mathcal{O}(1)^{\oplus d-3} \oplus \mathcal{O}(2)^{\oplus\binom{d-2}{2}}$
- *(c)* $E_\beta = \mathcal{O}(1)^{\oplus d-2} \oplus \mathcal{O}(2)$
- *(d)* $F_\beta = \mathcal{O}(2)^{\oplus \frac{d(d-3)}{2}}$

Proof. For item (a), note that the sequence

$$0 \to \mathcal{O}_{\mathbf{P}^1} \to \alpha_*\mathcal{O}_C \to E_\alpha^\vee \to 0$$

splits, and therefore $H^0(E_\alpha^\vee) = 0$. This means all summands of E_α are positive and add up to $d-1$. Therefore, all summands must have degree one.

For item (b), note that using the relative canonical factorization $\iota\colon R \to \mathbf{P}E$, we may think of R as lying inside $\mathbf{P}E$. The series $|\zeta - f|$ on $\mathbf{P}E$ restricts to the complete series $\mathcal{O}_R(d-2)$, and the morphism $q\colon \mathbf{P}E \to \mathbf{P}^{d-2}$ given by the

series $|\zeta - f|$ restricts to the embedding of R into $\mathbf{P}^{d-2}$ as a rational normal curve.

The rational normal curve R is contained in a $\binom{d-2}{2}$ dimensional space of quadrics. The divisor class of these quadrics, when pulled back along q, is $2\zeta - 2f$.

Recall the exact sequence on the target $\mathbf{P}^1$:

$$0 \to F_\alpha \to \operatorname{Sym}^2 E_\alpha \to \alpha_* \omega_\alpha^2 \to 0. \tag{10}$$

We twist by $\mathcal{O}(-2)$ and consider global sections, to get:

$$\begin{aligned} H^0(\mathbf{P}^1, F_\alpha(-2)) &= \ker(H^0(\mathbf{P}^1, \operatorname{Sym}^2 E_\alpha(-2)) \to H^0(\mathbf{P}^1, \alpha_*\omega_\alpha^2(-2))) \\ &= \ker(H^0(\mathbf{P}E_\alpha, 2\zeta - 2f) \to H^0(R, (2\zeta - 2f)|_R)) \\ &= \ker(H^0(\mathbf{P}^{d-2}, \mathcal{O}_{\mathbf{P}^{d-2}}(2)) \to H^0(R, \mathcal{O}_{\mathbf{P}^{d-2}}(2)|_R)) \end{aligned}$$

The previous paragraph along with projective normality of $R \subset \mathbf{P}^{d-2}$ imply that $h^0(F_\alpha(-2)) = \binom{d-2}{2}$. We conclude by noting that (10) shows that no degree of a summand of F_α may exceed 2, and the degree of F_α must be $(d-3)(d-1)$. This forces its splitting type to be $\mathcal{O}(1)^{\oplus d-3} \oplus \mathcal{O}(2)^{\oplus \binom{d-2}{2}}$.

For item (c), we note that all $d-1$ summands of E_β are positive, and their degrees sum to d. Therefore E_β must be as indicated.

Finally, for item (d), see that the pencil of sections of $\beta^*\mathcal{O}_{\mathbf{P}^1}(1)$ on X forms a two dimensional vector subspace of its complete linear system. The complete series gives an embedding of X into the projective space $\mathbf{P}^{d-1}$, and the map β is realized as projection from a general $(d-3)$-dimensional linear space $\Lambda \subset \mathbf{P}^{d-1}$ disjoint from X.

We note that $\mathbf{P}E$, as an abstract scroll, is isomorphic to $Y := \mathrm{Bl}_\Lambda \mathbf{P}^{d-1}$. The linear system $|\zeta - f|$ provides the map $f\colon \mathbf{P}E \to \mathbf{P}^{d-1}$. Furthermore, the linear system $|2\zeta - 2f|$ parametrizes the quadric hypersurfaces in $\mathbf{P}^{d-1}$. Therefore, $h^0(F_\beta(-2))$ is simply the vector space dimension of quadrics in $\mathbf{P}^{d-1}$ containing the elliptic normal curve $X \subset \mathbf{P}^{d-1}$, which is easily calculated to be $d(d-3)/2$.

Next we show that $h^0(F_\beta(-3)) = 0$. An element of the linear system $|2\zeta - 3f|$ corresponds to a quadric in $\mathbf{P}^{d-1}$ which splits off a hyperplane component Γ containing Λ. No such quadric can contain the elliptic normal curve X, by nondegeneracy of the curve. This means $h^0(F_\beta(-3)) = 0$.

Therefore, the largest degree summand of F_β is $\mathcal{O}(2)$. Since the degree of F_β, by sequence (10), is $d(d-3)$ we conclude that F_β must split as $\mathcal{O}(2)^{\oplus \frac{d(d-3)}{2}}$. □

Remark 2.8. Of course lemma 2.7 part (d) also follows from the much more general proposition 2.4.

3. Vector bundles on rational chains

In this section we find necessary and sufficient conditions for determining when a vector bundle on a chain of rational curves is balanced. We will eventually use these criteria to prove theorem 4.1.

Let $\mathbf{P} = P_1 \cup P_2 \cup \ldots \cup P_k$ be a chain of k rational curves P_i, and let V be a vector bundle on $\mathbf{P}$ of rank r. We let $V_i = V|_{P_i}$ be the restriction to the i-th component, and we denote by p_i the node $P_i \cap P_{i+1}$, for $i = 1, \ldots, k-1$.

Definition 3.1. A vector bundle V on a reduced curve C is *balanced* if

$$h^1(\operatorname{End} V) = 0.$$

Since the first order deformations of a vector bundle V are parametrized by $\operatorname{Ext}^1(V, V) = H^1(\operatorname{End} V)$, we see that a balanced vector bundle does not deform. Our goal is to determine necessary and sufficient criteria for a vector bundle V on the chain $\mathbf{P}$ to be balanced in terms of the "relative position" of the vector bundles V_i.

Our first easy observation is the following.

Lemma 3.2. *Let V be a vector bundle on $\mathbf{P}$ of rank r. If V is balanced, then every component V_i is a balanced vector bundle on P_i.*

Proof. Let $\nu \colon \coprod_i P_i \to \mathbf{P}$ be the total normalization map. Consider the exact sequence of $\mathcal{O}_{\mathbf{P}}$-sheaves

$$0 \to \operatorname{End} V \to \nu_* \nu^* (\operatorname{End} V) \to \oplus_i \operatorname{End}(k(p_i)^r) \to 0. \tag{11}$$

The last part of the long exact sequence of cohomology looks like

$$\ldots \to H^1(\mathbf{P}, \operatorname{End} V) \to H^1(\mathbf{P}, \nu_* \nu^* \operatorname{End} V) \to 0.$$

Since ν is finite, we can identify the vector space $H^1(\mathbf{P}, \nu_*\nu^* \operatorname{End} V)$ with $H^1(\coprod P_i, \nu^* \operatorname{End} V) = \oplus_i H^1(P_i, \operatorname{End} V_i)$. Therefore, if $H^1(\mathbf{P}, \operatorname{End} V) = 0$, then $\oplus_i H^1(P_i, \operatorname{End} V_i) = 0$, which proves the lemma. $\square$

We now assume that each component V_i is balanced. Consider the long exact sequence of cohomology groups associated to (11):

$$\ldots \to \oplus_i H^0(P_i, \operatorname{End} V_i) \to \oplus_i \operatorname{End}(k(p_i)^r) \to H^1(P, \operatorname{End} V) \to 0 \tag{12}$$

This sequence shows that $h^1(P, \operatorname{End} V) = 0$ if and only if the map

$$D\colon \oplus_i H^0(P_i, \operatorname{End} V_i) \to \oplus_i \operatorname{End}(k(p_i)^r)$$

is surjective. The map D is described as follows. Let $\infty_i \in P_i$, $0_{i+1} \in P_{i+1}$ denote the two pre images of the node p_i in the normalization $\coprod P_i$. For any $(M_1, \ldots, M_k) \in \oplus_i H^0(P_i, \operatorname{End} V_i)$,

$$D(M_1, \ldots, M_k) = (M_1|_{\infty_1} - M_2|_{0_2}, \ldots, M_{k-1}|_{\infty_{k-1}} - M_k|_{0_k}).$$

In other words, since the structure sheaves $k(\infty_i)$ and $k(0_{i+1})$ are naturally identified with $k(p_i)$ via the normalization map ν, we may compare the

restriction of any section $M_i \in H^0(P_i, \operatorname{End} V_i)$ at ∞_i with the restriction of any section $M_{i+1} \in H^0(P_{i+1}, \operatorname{End} V_{i+1})$ at 0_{i+1}. For this reason, we call D the *difference* map. (We are using the letter M to emphasize that we are thinking of these sections as $r \times r$ matrices.)

Recall that the first order deformation space of a vector bundle V is naturally identified with the vector space $H^1(\operatorname{End} V)$. Consider the connecting homomorphism

$$\oplus_i \operatorname{End}(k(p_i)^r) \to H^1(P, \operatorname{End} V).$$

There is a very simple way of interpreting this map in terms of the deformations of V. A vector bundle V on $\mathbf{P}$ is equivalent to the data of the k vector bundles $(V_1, \ldots, V_k)$ and gluing maps $(g_1, \ldots, g_{k-1})$, where $g_i \colon V_i|_{\infty_i} \to V_{i+1}|_{0_{i+1}}$ are invertible linear maps. We can deform this data by simply varying the gluing maps, viewed as invertible $r \times r$ matrices. The tangent space to $GL_r(k)$ is the space $M_{r\times r}(k)$ of $r \times r$ matrices, and the resulting $k-1$-tuple of of elements in $M_{r\times r}$ correspond to components in the direct sum $\oplus_i \operatorname{End}(k(p_i)^r)$. Therefore, by considering different gluing data, we think of the condition of being balanced as a condition about the "relative position" of the V_i being general. We will make this more precise in the next subsection.

3.1. **Directrices and Filtrations.** A balanced bundle on $\mathbf{P}^1$ has a canonically defined *directrix* subbundle.

Definition 3.3. Let $V = \mathcal{O}(m)^{\oplus a} \oplus \mathcal{O}(m+1)^{\oplus b}$ be a balanced vector bundle on $\mathbf{P}^1$. Then the *directrix* subbundle W is the summand $\mathcal{O}(m+1)^{\oplus b}$.

Now let us mark two points 0 and ∞ on $\mathbf{P}^1$, and let $[s : t]$ be homogeneous coordinates on $\mathbf{P}^1$ with $0 = [0 : 1]$ and $\infty = [1 : 0]$. Suppose V is a balanced vector bundle on $\mathbf{P}^1$ and $W \subset V$ its directrix subbundle. Of course, there is no canonical identification of $V|_0$ with $V|_\infty$, since the vector bundle is not trivial. However, since $W = \mathcal{O}(m+1)^{\oplus b}$, the subbundle $\mathbf{P}W \subset \mathbf{P}V$ *is* trivial. Therefore, the subspace $W|_0 \subset V|_0$ naturally corresponds to the subspace $W|_\infty \subset V|_\infty$, *as a subspace*. Similarly, any linear subspace $L|_0 \subset W|_0$ corresponds to a unique subspace $L_\infty \subset W_\infty$.

By the same token, since $V/W = \mathcal{O}(m)^{\oplus a}$, we see that any subspace $N|_0 \subset V/W|_0$ can be naturally identified with a subspace $N|_\infty \subset V/W|_\infty$. To summarize we have established the following.

Lemma 3.4. *Let $W \subset V$ be as above. Then subspaces of $W|_0$ are in natural one to one correspondence with subspaces of $W|_\infty$. Furthermore, intermediate subspaces $W|_0 \subset N_0 \subset V|_0$ are in natural one to one correspondence with intermediate subspaces $W|_\infty \subset N_\infty \subset V|_\infty$.*

Definition 3.5. Let $F^\bullet$ be an increasing filtration of a vector space A, and let $B \subset A$ be a subspace. We say that $F^\bullet$ *contains* B if $B = F^i$ for some i.

Corollary 3.6. *The filtrations $F_0^\bullet$ of $V|_0$ containing $W|_0$ are in natural one to one correspondence with the filtrations $F_\infty^\bullet$ of $V|_\infty$ containing $W|_\infty$.*

We let

$$r\colon \left\{\text{filt. } F_0^\bullet \text{ of } V|_0 \text{ containing } W|_0 \right\} \to \left\{\text{filt. } F_\infty^\bullet \text{ of } V|_\infty \text{ containing } W|_\infty\right\}$$

denote the natural correspondence, and let l denote its inverse. We call $r(F_0^\bullet)$ the *right transport* of $F_0^\bullet$, and $l(F_\infty^\bullet)$ the *left transport* of $F_\infty^\bullet$

For a balanced vector bundle V with directrix $W \subset V$, we define

$$G^\bullet(V) := \{0 \subset W \subset V\}$$

to be the *directrix flag* of V. Furthermore, we let $G^\bullet(V|_0)$ and $G^\bullet(V|_\infty)$ denote the respective flags in $V|_0$ and $V|_\infty$.

Definition 3.7. Let $F^\bullet = \{0 = F^0 \subset F^1 \subset \ldots \subset F^N = A\}$ and $G^\bullet = \{0 \subset B \subset A\}$ be two filtrations of a k-vector space A. Then the *modification of* $F^\bullet$ *by* $G^\bullet$, denoted $F \vee G$, is the flag with elements $(F \vee G)^i = F^i \cap B$ for $i \leq N$, and $(F \vee G)^j = B + F^{j-N}$ for $j \geq N$.

3.1.1. *The two natural filtrations at a node.* Now let $\mathbf{P} := P_1 \cup P_2 \cup \ldots \cup P_k$ be a chain of $\mathbf{P}^1$'s. Recall that we let 0_i and ∞_i denote marked points 0 and ∞ on the respective P_i, and p_j be the node joining P_j with P_{j+1}, so that $p_j = \infty_j = 0_{j+1}$.

Let V be a rank r vector bundle on $\mathbf{P}$. We continue to assume that each V_i is balanced, and let $W_i \subset V_i$ be the i-th directrix subbundle.

At each node p_j there are two natural filtrations of $V|_{p_j}$, called the *left filtration* $L_j^\bullet$ and the *right filtration* $R_j^\bullet$, which we describe inductively. Let $L_1^\bullet = G^\bullet(V_1|_0)$, and inductively define $L_j^\bullet$ as the filtration given by

$$L_j^\bullet := r\left(L_{j-1}^\bullet \wedge G^\bullet(V_j|_0)\right).$$

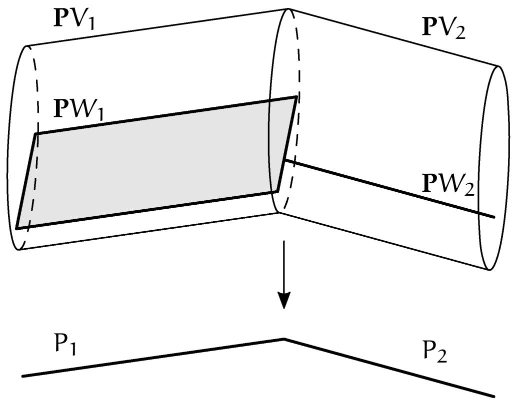
In words, we sequentially right transport and modify the directrix flags of the vector bundles V_i, beginning at V_1 and ending at V_j. The filtration $R_j^\bullet$ is defined similarly, except we begin at V_k and then sequentially left transport and modify the directrix subbundles until we reach V_{j+1}:

$$R_j^\bullet = l\left(R_{j+1}^\bullet \wedge G^\bullet(V_{j+1}|_\infty)\right)$$

The essential point is that the collection of filtrations $L_j^\bullet$ and $R_j^\bullet$ completely captures the "relative position" of the system of directrices $W_i \subset V_i$.

Definition 3.8. We say that a vector bundle V on $\mathbf{P} = P_1 \cup \ldots \cup P_k$ as above *has transverse directrices* if the filtrations $L_j^\bullet$ and $R_j^\bullet$ meet transversely for each node p_j.

See fig. 2 for examples of bundles whose directrices fail to meet transversely. We can now state the theorem which characterizes balanced vector bundles on chains of $\mathbf{P}^1$'s.



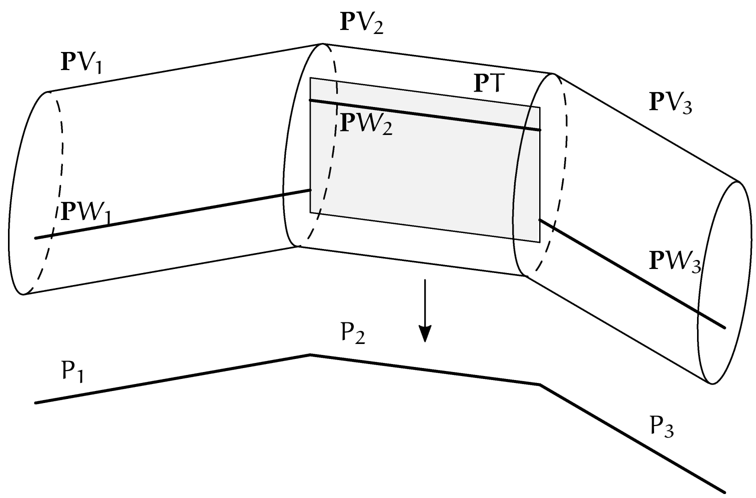


FIGURE 2. Some bundles whose directrices fail to meet transversely.
In the first diagram, the codimension 2 directrix $\mathbf{P}W_2$ of $\mathbf{P}V_2$ meets the codimension 1 directrix $\mathbf{P}W_1$ of $\mathbf{P}V_1$ over the node $P_1 \cap P_2$, even though $\mathbf{P}V$ is a $\mathbf{P}^2$-bundle.
In the second diagram, although adjacent directrices meet transversely, if we transport the flag above node $P_1 \cap P_2$ to node $P_2 \cap P_3$ via the minimal subbundle $\mathbf{P}T$, we get non-transverse intersection with $\mathbf{P}W_3$.

Theorem 3.9. *Let* V *be a vector bundle on* $\mathbf{P} = P_1 \cup \ldots \cup P_k$*. Then* $h^1(\mathrm{End}\, V) = 0$ *if and only if*

(1) Each V_i *is balanced, for* $i = 1, \ldots, k$*, and*
(2) V *has transverse directrices.*

The remainder of this section will be spent proving theorem 3.9.

Lemma 3.2 already states that if V is balanced, then each V_i is balanced as well. The following proposition shows that V also has to have transverse directrices.

Proposition 3.10. *Let V be a vector bundle on $\mathbf{P}$ such that V_i are balanced for all i. Furthermore, suppose there is a node p_j where $L_j^\bullet$ and $R_j^\bullet$ fail to intersect transversely. Then V is not balanced, i.e. V admits a non trivial first order deformation.*

Proof of proposition 3.10. We can produce non-trivial deformations of V by modifying the gluing over p_j, in a way that the filtrations $L_j^\bullet$ and $R_j^\bullet$ become transverse. While it is clear that there will be non-trivial deformations over a DVR, we need to check that the restriction to $\mathbf{D} = \operatorname{Spec} k[\varepsilon]/(\varepsilon^2)$ is still non-trivial.

We get our hands on the first order deformations from the boundary map

$$\delta\colon \oplus_i H^0(\operatorname{End} k(p_i)^r) \to H^1(\operatorname{End} V).$$

Given $M = (M_1, \dots, M_{k-1}) \in \oplus_i H^0(\operatorname{End} k(p_i)^r)$, we construct the corresponding first order deformation $\delta(M)$ of V as follows. Take the (trivially deformed) bundle $V_i \times \mathbf{D}$ on each component $P_i \times \mathbf{D}$, and glue these vector bundles using the map

$$f_i + \varepsilon M_i\colon (V_i|_\infty \times \mathbf{D}) \to (V_{i+1}|_0 \times \mathbf{D})$$

where $f_i\colon V_i|_\infty \to V_{i+1}|_0$ are the original gluing maps for the vector bundle V on P.

We claim that a generic choice of M makes the deformation obtained by the gluings above nontrivial. As a matter of fact, we can even take $M_i = 0$ for $i \neq j$, as long as M_j *separates* the flags $L_j^\bullet$ and $R_j^\bullet$ in the following sense.

Definition 3.11. Let V be an r-dimensional k-vector space, and let $V[\varepsilon]$ be the module $V \otimes_k k[\varepsilon]/(\varepsilon^2)$. Suppose A and B are subspaces of V which fail to meet properly. Let M be an endomorphism of V and consider the invertible module map

$$f_M := \mathrm{Id} + \varepsilon M\colon V[\varepsilon] \to V[\varepsilon].$$

We say that M *separates* A and B if the submodule $A[\varepsilon] \cap f_M(B[\varepsilon])$ is not flat as a $k[\varepsilon]/(\varepsilon^2)$-module.

Note that if M_j separates the flags $L_j^\bullet$ and $R_j^\bullet$, then the deformation is not trivial. We just have to show that a generic choice of M_j does separate these flags. The following lemma assures that this is the case.

Lemma 3.12. *Let V, A, and B be as in definition 3.11. Then there exists $M \in \operatorname{End} V$ which separates A and B. Furthermore, the general choice of M separates A and B.*

Proof of lemma 3.12. Note that flat and free are the same notion for the local Artinian ring $k[\varepsilon]/(\varepsilon^2)$. Furthermore, a $k[\varepsilon]/(\varepsilon^2)$-module N is flat if and only if the multiplication map $\times\varepsilon\colon (N/\varepsilon N) \to N$ is injective.

We now choose a basis $\{v_1, \dots, v_r\}$ for V such that $A = \langle v_1, \dots, v_a\rangle$ and $B = \langle v_{m+1}, \dots, v_{m+b}\rangle$, and assume that $m + b < r$ and $m + 1 \le a$, so that A and B do not meet properly. Now consider the endomorphism $M \in \operatorname{End} V$ which is 0 on all v_i except sends v_{m+1} to v_r. Then $f_M(B[\varepsilon]) = \langle v_{m+1} + \varepsilon v_r, v_{m+2}, \dots, v_{m+b}\rangle \subset V[\varepsilon]$. The element εv_{m+1} is annihilated by ε, but is non-zero in the quotient module $A[\varepsilon] \cap f_M(B[\varepsilon])/\varepsilon(A[\varepsilon] \cap f_M(B[\varepsilon]))$. Hence, $A[\varepsilon] \cap f_M(B[\varepsilon])$ is not flat. □

Lemma 3.12 now implies proposition 3.10. □

Proposition 3.10 tells us conditions (1) and (2) are necessary in theorem 3.9. Now we must prove sufficiency. Specifically, we must show that if V has transverse directrices, then the difference map

$$D\colon \oplus_i H^0(P_i, \operatorname{End} V_i) \to \oplus_i \operatorname{End}(k(p_i)^r)$$

is surjective. For $M \in \oplus_i H^0(P_i, \operatorname{End} V_i)$, denote by $D_i(M)$ the i-th component of $D(M) \in \oplus_i \operatorname{End}(k(p_i)^r)$. It is enough to show that for each j, we can choose M such that $D_i(M) = 0$ for all $i \neq j$, while $D_j(M) \in \operatorname{End}(k(p_j)^r)$ is arbitrary.

We will first choose M_i for $i < j$ such that $D_i(M) = 0$ for all $i < j$, and see what constraints we have on $M_{j-1}|_\infty$. We will then do the same from the other side, starting at M_{k-1} and going down to M_j, and then conclude that the difference $D_j(M) = M_{j-1}|_\infty - M_j|_0$ can be made to be arbitrary.

To express the constraints on $M_{j-1}|_\infty$, let us introduce the following notation:

Definition 3.13. We say that an endomorphism $M \in \operatorname{End}(V)$ of a vector space V *respects the flag* $0 \subset F_1 \subset F_2 \subset \ldots \subset F_r \subset V$ if M preserves the F_i as subspaces of V.

We now can state the following.

Lemma 3.14. *We can choose* $M_i \in H^0(P_i, \operatorname{End} V_i)$ *for* $i < j$ *such that* $D_i(M) = 0$ *for* $i < j$, *and* $M_{j-1}|_\infty \in \operatorname{End}(k(p_j)^r)$ *is an arbitrary endomorphism that preserves the left flag* $L_j^\bullet$.

Of course the analogous result is true if we started from the right: we can choose $M_i \in H^0(P_i, \operatorname{End} V_i)$ for $i \ge j$ such that $D_i(M) = 0$ for $i > j$ and $M_j|_0 \in H^0(P_i, \operatorname{End} V_i)$ is an arbitrary endomorphism preserving the *right* flag $R_j^\bullet$. Now we only have to assure that we can arrange the difference $M_{j-1}|_\infty - M_j|_0$ to be arbitrary. The following lemma does that.

Lemma 3.15. *If* $F^\bullet$ *and* $G^\bullet$ *are transverse flags of a vector space* V, *then any endomorphism of* V *can be written as a difference of an endomorphism respecting the flag* $F^\bullet$, *and one respecting* $G^\bullet$.

Proof of lemma 3.15. Without loss of generality, we may assume that $F^\bullet$ and $G^\bullet$ are complete flags. Indeed, if they are not, just choose generic finer complete flags containing them, and the problem only becomes more restrictive.

Now pick a basis $v_1, \dots, v_n$ of V such that $\langle v_i \rangle = F^i \cap G^{n-i-1}$. In this basis, endomorphisms preserving $F^\bullet$ are upper triangular matrices, and the ones respecting $G^\bullet$ are the lower triangular ones. And any matrix can be written as a difference of two such matrices. □

To complete the proof of theorem 3.9, we only need to prove lemma 3.14.

Proof of lemma 3.14. We proceed by induction on the length of the chain $\mathbf{P}$. If there is only one component, the claim states that for any endomorphism $\overline{M} : V_1|_\infty \to V_1|_\infty$ which sends the restriction of the directrix $W|_\infty$ to itself, we can find $M \in H^0(\mathrm{End}(V_1))$ such that $M|_\infty = \overline{M}$.

Fix a splitting $V_1 = \mathcal{O}(m)^a \oplus \mathcal{O}(m+1)^b$. Then $\mathrm{End}(V_1) = \mathcal{O}^{a^2} \oplus \mathcal{O}(1)^{ab} \oplus \mathcal{O}(-1)^{ab} \oplus \mathcal{O}^{b^2}$, and we can realize this splitting more naturally by realizing an element of $\mathrm{End}\, V_1$ as an $(a+b) \times (a+b)$ matrix block matrix. For example, for $a = 2, b = 3$, we have

$$\mathrm{End}\, V_1 = \left[\begin{array}{cc|ccc} \mathcal{O} & \mathcal{O} & \mathcal{O}(1) & \mathcal{O}(1) & \mathcal{O}(1) \\ \mathcal{O} & \mathcal{O} & \mathcal{O}(1) & \mathcal{O}(1) & \mathcal{O}(1) \\ \hline \mathcal{O}(-1) & \mathcal{O}(-1) & \mathcal{O} & \mathcal{O} & \mathcal{O} \\ \mathcal{O}(-1) & \mathcal{O}(-1) & \mathcal{O} & \mathcal{O} & \mathcal{O} \\ \mathcal{O}(-1) & \mathcal{O}(-1) & \mathcal{O} & \mathcal{O} & \mathcal{O} \end{array}\right]$$

The global sections of $\mathrm{End}(V_1)$ restrict to an arbitrary block upper triangular matrix at ∞. And being block upper triangular exactly means that the directrix is preserved. This proves the case for one component.

For the induction step, we are given an endomorphism $\overline{M}$ of $V_{j-1}|_\infty$ preserving $L_j^\bullet$, and we want to find $M \in H^0(\mathrm{End}(V_{j-1})$ such that

- $M|_\infty = \overline{M}$, and
- $M|_0$ is an endomorphism of $V_{j-1}|_0$ respecting $L_{j-1}^\bullet$.

This is enough for our purposes, because by induction we can choose global sections of $\mathrm{End}(V_i)$ for $i < j-1$ such that all differences D_i are zero, for $i < j-1$, and produce $M|_0$ over the $(j-1)$-th node.

We will choose an appropriate splitting of V_{j-1} so that endomorphisms preserving the flags $L_{j-1}^\bullet$ and $L_j^\bullet$ become block matrices.

Definition 3.16. An ordered basis $v_1, \dots, v_n$ *generates the flag* $F^\bullet$ if each F^i is the span of $v_1, v_2, \dots, v_{n_i}$ for some n_i.

Claim 3.17. *We may choose an ordered basis* $v_1, \dots, v_n$ *of* $V_{j-1}|_0$ *generating* $L_{j-1}^\bullet$ *and such that* $W_{j-1}|_0 = \langle v_k, v_{k+1}, \dots, v_n \rangle$.

Proof of claim 3.17. We use the same idea as in the proof of lemma 3.15. Complete $L_{j-1}^\bullet$ and $G^\bullet = (0 \subset W_{j-1}|_0 \subset V_{j-1}|_0)$ to generic complete flags $\widehat{L}_{j-1}^\bullet$ and

$\widehat{G}^\bullet$. Since $G^\bullet$ is transverse to $L^\bullet_{j-1}$ by assumption, so is $\widehat{L}^\bullet_{j-1}$ and $\widehat{G}^\bullet$. Choose v_i such that $\langle v_i \rangle = \widehat{L}^i_{j-1} \cap \widehat{G}^{n-i-1}$. □

Note that the modification $L^\bullet_{j-1} \wedge (0 \subset W_{j-1}|_0 \subset V_{j-1}|_0)$ is generated by the ordered basis $(v_k, v_{k+1}, \ldots, v_n, v_1, v_2, \ldots, v_{k-1})$.

Now pick a splitting of V_{j-1} that restricts to the basis $v_1, \ldots, v_n$ over zero. Let us write the matrix $M \in H^0(\mathrm{End}(V_{j-1}))$ in this basis. For convenience, we will use block notation, separating the vectors generating W (that is, $v_k, \ldots, v_n$), from the remaining $v_1, \ldots, v_{k-1}$.

$$M = \left[\begin{array}{c|c} M(V/W, V/W) & M(W, V/W) \\ \hline M(V/W, W) & M(W, W) \end{array}\right]$$

We want $M|_0$ to respect the flag $L^\bullet_{j-1}$, and $M|_\infty$ to be the given matrix $\overline{M}$ respecting the L_j flag. In this block notation this translates into the following conditions:

(1) The matrix $M(V/W, V/W)$ is upper block triangular when restricted to both zero and infinity (for the same block shapes). The entries of $M(V/W, V/W)$ are valued in $\mathcal{O}_{\mathbf{P}^1}$, so we must take the constant matrix specified by $\overline{M}$.
(2) The same applies for the block $M(W, W)$: we want it to be upper block triangular at zero, with the restriction at infinity specified by $\overline{M}$. Since the matrix $M(W, W)$ is valued in $\mathcal{O}_{\mathbf{P}^1}$, we must use a constant matrix.
(3) The entries of $M(W, V/W)$ are valued in $\mathcal{O}_{\mathbf{P}^1}(-1)$, so it has to be the zero matrix. This poses no problems, because at infinity the corresponding block of $\overline{M}$ is automatically zero to begin with.
(4) Finally, the block $M(V/W, W)$ must restrict to a specified matrix over infinity (given by $\overline{M}$), but has to vanish over zero. This can be achieved, because all the entries are valued in $\mathcal{O}_{\mathbf{P}^1}(1)$, so we may take linear functions interpolating between arbitrary values at zero and infinity.

This completes the proof lemma 3.14, and hence by induction the proof of theorem 3.9. □

4. Balanced F-bundles and a maximal rank problem

The goal of this section is to translate the Main Theorem into a degree 2 maximal rank problem for "maximally connected chains" in $\mathbf{P}^r$. This will in fact allow us to establish the following slightly stronger theorem.

Theorem 4.1. *The general cover* $[\alpha\colon C \to \mathbf{P}^1] \in \mathcal{H}_{d,g}$ *has a balanced bundle of quadrics* F *whenever* g *can be written as* $(a-1)(d-1) + bd$*, for integers* $a, b \geq 0$.

The Main Theorem follows from theorem 4.1 and the following lemma.

Lemma 4.2. *Any integer* $g \geq (d-3)(d-1)$ *can be written as* $(a-1)(d-1)+bd$ *for integers* $a, b \geq 0$.

Proof. We may write $g = q(d-1) + r$ for

$$q \geq d-3 \text{ and } d-2 \geq r \geq 0$$

Setting $b = r \geq 0$ and $a = q - r + 1 \geq 0$, we have

$$(a-1)(d-1) + bd = (q-r)(d-1) + rd = q(d-1) + r = g$$

as we wanted. □

We prove theorem 4.1 by first observing that being balanced is an open condition. Hence, it is enough to exhibit a single (admissible) cover with balanced F-bundle. We will use theorem 3.9 to translate this condition to a version of the maximal rank problem, specifically theorem 5.1.

The specific admissible cover we will consider arises from the following construction. Given:

- degree d, genus g_i simply branched covers $[\alpha_i \colon C_i \to \mathbf{P}^1] \in \mathcal{H}_{d,g_i}$ unramified over 0 and ∞, for $i = 1, \dots, n$, and
- bijections $\varphi_i : \alpha_i^{-1}(\infty) \to \alpha_{i+1}^{-1}(0)$, for $i = 1, \dots, n-1$

we may construct the nodal curve X, with irreducible components $C_1, \dots, C_n$, obtained by identifying each $p \in \alpha_i^{-1}(0) \subset C_i$ with $\varphi(p) \in \alpha_{i+1}^{-1}(\infty) \subset C_{i+1}$ via φ_i, for all i. (See fig. 3.) The curve X has $d(n-1)$ nodes in total, and admits a map

$$\alpha \colon X \to \mathbf{P}_n$$

to a chain of n $\mathbf{P}^1$'s, which we denote by $\mathbf{P}_n$. The map α is an admissible cover in the sense of [HM82].

Denote by $\Sigma(d; g_1, \dots, g_n)$ the parameter space of admissible covers obtained by this procedure. Note that

$$g = p_a(X) = 1 + \sum_i (g_i - 1) + d(n-1). \tag{13}$$

Lemma 4.3. *The space* $\Sigma(d; g_1, \dots, g_n)$ *is irreducible.*

Proof. Consider the forgetful finite map from $g : \Sigma \to \mathcal{H}_{d,g_1} \times \mathcal{H}_{d,g_2} \times \dots \times \mathcal{H}_{d,g_n}$. The target is irreducible and smooth, so we only have to show that the monodromy on the fibers is transitive. The fibers correspond to choosing different systems of bijections $\{\varphi_i\}$. But each simply branched cover $\alpha_i : C_i \to \mathbf{P}^1$ has full monodromy, which in turn induces a transitive action on the set of systems of bijections $\{\varphi_i\}$. □

From now on, we focus our attention on proving the following proposition, which implies theorem 4.1.

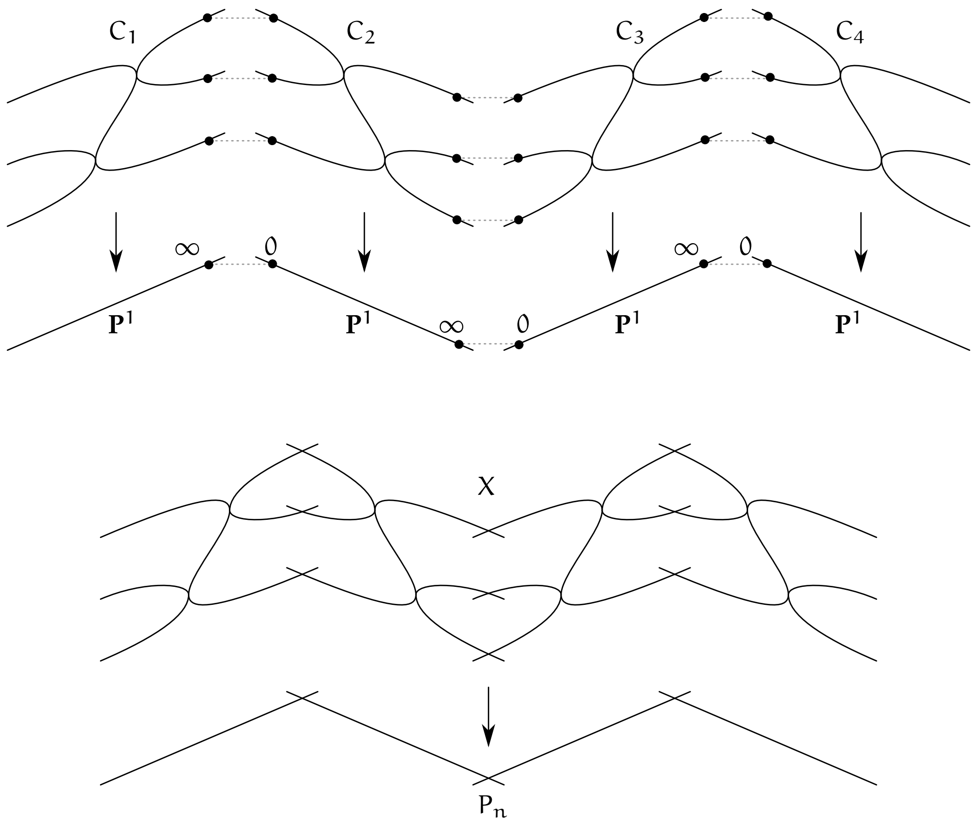


FIGURE 3. An admissible cover $[X \to \mathbf{P}_n]$ in $\Sigma(d; g_1, \ldots, g_n)$ for $n = 4, d = 3$.

Proposition 4.4. *If* $g_1 = \ldots = g_a = 0$ *and* $g_{a+1} = \ldots = g_{a+b} = 1$, *then the general admissible cover* $\alpha\colon X \to \mathbf{P}_{a+b}$ *in* $\Sigma(d; g_1, \ldots, g_{a+b})$ *has balanced* F*-bundle.*

Note that in the context of the proposition above, eq. (13) becomes

$$\begin{aligned} g &= 1 + \sum_i (g_i - 1) + d(a + b - 1) \\ &= 1 + a(0 - 1) + b(1 - 1) + d(a + b - 1) \\ &= (a - 1)(d - 1) + bd. \end{aligned}$$

We start the proof of proposition 4.4 by reducing to the case where all components are rational.

Lemma 4.5. *To prove proposition 4.4, it suffices to prove the case* $b = 0$*.*

Proof. Let $\alpha : X \to \mathbf{P}_{a+b}$ be a general cover in $\Sigma(d; 0, ..., 0, 1, ...1)$. Split it into the a rational components $X_R \to \mathbf{P}_a$ and the b genus one components $X_E \to \mathbf{P}_b$.

The forgetful map $\Sigma(d; 0, ..., 0, 1, ...1) \to \Sigma(d; 0, ..., 0)$ (which sends the cover $X \to \mathbf{P}_{a+b}$ to $X_R \to \mathbf{P}_a$) is dominant. Hence, if we assume proposition 4.4 is true when $b = 0$, we may assume the F-bundle of $X_R \to \mathbf{P}_a$ is balanced.

Moreover, by item (d) of lemma 2.7, the F-bundle of $X_E \to \mathbf{P}_b$ is not only balanced, it is *trivial*, up to a twist by a line bundle. Hence, the F-bundle of $\alpha : X \to \mathbf{P}_{a+b}$—which is obtained by gluing the (balanced) F-bundle of $X_R \to \mathbf{P}_a$ with the (trivial, up to twist) F-bundle of $X_E \to \mathbf{P}_b$—is balanced as well. □

Given this lemma, we need only consider the case $g_1 = \ldots = g_a = 0$. So we simplify notation by setting

$$\Sigma_{d,a} := \Sigma(d; 0, 0, \ldots, 0).$$

Let $\alpha : X \to \mathbf{P}_a$ be a general cover in $\Sigma_{d,a}$. We want to show that its F-bundle is balanced. Each component of X is rational—we will denote the components by $R_1, R_2, \ldots, R_a$.

From lemma 2.7, the Tschirnhausen bundle E_{α_i} for each component R_i is trivial up to a twist by a line bundle. Hence the Tschirnhausen bundle E for the entire cover α: $X \to \mathbf{P}_a$ is trivial up to a twist as well, which implies $\mathbf{P}(E) \simeq \mathbf{P}^{d-2} \times \mathbf{P}_a$. Composing the relative canonical embedding $X \hookrightarrow \mathbf{P}(E) = \mathbf{P}^{d-2} \times \mathbf{P}_a$ with the projection onto the first factor gives us a map $X \to \mathbf{P}^{d-2}$. As in the proof of lemma 2.7, this map is easily seen to be an embedding, and each component R_i maps to a rational normal curve in $\mathbf{P}^{d-2}$. The resulting embedded curve $X \subset \mathbf{P}^{d-2}$ is the main object of our investigation moving forward, so we make the following definition:

Definition 4.6. A *maximally connected chain of length* n in $\mathbf{P}^r$ is a reducible nodal curve $X = R_1 \cup \ldots R_n \subset \mathbf{P}^r$ with n components, which are called *links*. Each link R_i is a non-degenerate, degree r, nodal, connected curve, and consecutive links $R_i \cap R_{i+1}$ meet in $r + 2$ points; otherwise there are no further intersections between components. The points $R_i \cap R_{i+1}$ will be called *anchor points*. See fig. 4.

Let $MC_{r,n}$ denote the quasi-projective variety parametrizing all maximally connected chains of length n in $\mathbf{P}^r$.

Remark 4.7. For a generic maximally connected chain, each link R_i is a smooth rational normal curve. We will eventually need to consider degenerate maximally connected chains where the R_i become singular.

[1] Strictly speaking, our definition definition 4.6 does not allow for maximally connected chains in $\mathbf{P}^2$ because pairs of non-consecutive links will meet. However, for illustration

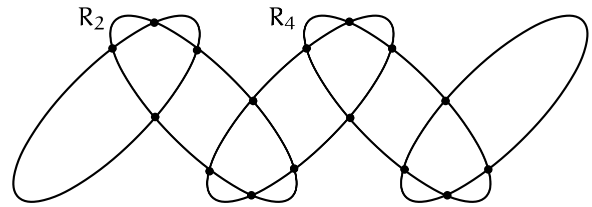


FIGURE 4. An illustration[1] of a maximally connected chain in $\mathbf{P}^2$.

Lemma 4.8. *The space* $MC_{r,n}$ *is irreducible of dimension* $(r-1)(r+3)+(n-1)(2r+1)$.

Proof. We use induction on n. For $n=1$, we are simply parametrizing rational normal curves in $\mathbf{P}^r$ (possibly nodal), and their parameter space is irreducible of dimension $(r-1)(r+3)$, as we wanted to show.

For the induction step, notice that the space $MC_{r,n}$ admits a dominant forgetful morphism to $MC_{r,n-1}$ by forgetting the last component. It is enough to show that the fibers are irreducible of dimension $2r+1$. The fiber over $R_1 \cup \dots R_{n-1} \in MC_{r,n-1}$ is an open subset of the scheme of rational normal curves R_n meeting R_{n-1} in $r+2$ points. This is irreducible, since upon fixing the $r+2$ points of intersection, the space of rational normal curves through them is isomorphic to the moduli space $\overline{\mathcal{M}}_{r+2,0}$, which has dimension $r-1$. (This is the *Kapranov model* of $\overline{\mathcal{M}}_{r+2,0}$ as in [Kap93].) In total, the fiber of the forgetful map has dimension $(r+2)+(r-1)=2r+1$, which allows us to conclude the lemma. $\square$

By projecting an admissible cover of type $\alpha : X \to \mathbf{P}_a$ in its relative canonical embedding $X \hookrightarrow \mathbf{P}^{d-2} \times \mathbf{P}_a$ to $\mathbf{P}^{d-2}$, we obtain a maximally connected chain of length a in $\mathbf{P}^{d-2}$. That is, there is a natural map $f : \Sigma_{d,a} \to MC_{d-2,a}$. Conversely, we claim that a generic maximally connected chain can be realized as such a projection of an admissible cover in $\Sigma_{d,a}$.

Lemma 4.9. *The map* $f : \Sigma_{d,a} \to MC_{d-2,a}$ *is dominant.*

Proof. Let $X \subset \mathbf{P}^{d-2}$ be a general maximally connected chain of length n, and let $R_1, \dots, R_a$ be its components. For $i = 2, \dots, a-1$, the rational curve R_i has a distinguished pencil of divisors on $\mathcal{O}_{R_i}(d)$, namely the pencil spanned by the two sets of anchor points $R_i \cap R_{i-1}$ and $R_i \cap R_{i+1}$. Let $0 \in \mathbf{P}^1$ correspond to the former, and let $\infty \in \mathbf{P}^1$ correspond to the latter. Now, for R_1 (and R_a), simply choose a basepoint free pencil of sections of $\mathcal{O}_{R_1}(d)$ (resp. $\mathcal{O}_{R_a}$) containing $R_1 \cap R_2$ (resp. $R_{a-1} \cap R_a$). The data of the curves R_i along with the pencils of degree d divisors on each gives rise to an admissible cover $\alpha \colon X \to \mathbf{P}_a$. The map $X \to \mathbf{P}^{d-2} \times \mathbf{P}_a$ is the relative canonical embedding

purposes the real points of a chain of conics make a more informative and legible picture than a chain of twisted cubics, meeting pairwise at 5 points.

for α, because each fiber $X_t \subset \mathbf{P}^{d-2}$ is a length d arithmetically Gorenstein subscheme. Therefore, $f(\alpha)$ is the original maximally connected chain $X \subset \mathbf{P}^{d-2}$, as we wanted to show. □

Two properties of maximally connected chains will be important for us.

Definition 4.10. A maximally connected chain $X = R_1 \cup \ldots \cup R_n \subset \mathbf{P}^r$ is *quadric-generic* if for any subchain $Y = R_i \cup R_{i+1} \cup \ldots \cup R_{i+j}$, the restriction map

$$H^0(\mathbf{P}^r, \mathcal{O}_{\mathbf{P}^r}(2)) \to H^0(Y, \mathcal{O}_Y(2))$$

has maximal rank.

Definition 4.11. Let $X = R_1 \cup \ldots \cup R_n \subset \mathbf{P}^r$ be a maximally connected chain. The residual intersection with R_{i+1} of a quadric Q containing a link R_i is

$$\mathrm{res}_{R_{i+1}}(Q) = Q \cap R_{i+1} - R_i \cap R_{i+1}$$

That is, the $r-2$ points of intersection of Q with R_{i+1} besides the $r+2$ anchor points $R_i \cap R_{i+1}$.

Definition 4.12. A maximally connected chain $R_1 \cup \ldots \cup R_n \subset \mathbf{P}^r$ has *transverse residues* if either

- r is even, or
- r is odd and for any subchain Y of length $r+2$, there are no quadrics $Q_{\text{left}}, Q_{\text{right}}$ satisfying

$$Y_{\text{left}} \subset Q_{\text{left}},\ Y_{\text{right}} \subset Q_{\text{right}} \text{ and } \mathrm{res}_{R_{\text{middle}}}(Q_{\text{left}}) = \mathrm{res}_{R_{\text{middle}}}(Q_{\text{right}})$$

  where
  - Y_{left} be the first $\frac{r+1}{2}$ links of Y,
  - R_{middle} the middle rational curve, and
  - Y_{right} the remaining $\frac{r+1}{2}$ links.

Remark 4.13. While contrived at first sight, having transverse residues simply reflects the expectation that the naive dimension count goes through. Indeed, the pair of quadrics Q_{left} and Q_{right} has to satisfy

- $(r+2) + (r-1)\frac{r+1}{2}$ conditions for $Y_{\text{left}} \subset Q_{\text{left}}$, plus
- $(r+2) + (r-1)\frac{r+1}{2}$ conditions for $Y_{\text{right}} \subset Q_{\text{right}}$, plus
- $r-2$ conditions for $\mathrm{res}_{R_{\text{middle}}}(Q_{\text{left}}) = \mathrm{res}_{R_{\text{middle}}}(Q_{\text{right}})$.

We get a total of

$$2 \times \left(r + 2 + (r-1)\frac{r+1}{2}\right) + r - 2 = r^2 + 3r + 1 = 2 \times \left(\binom{r+2}{2} - 1\right) + 1$$

conditions, which exceeds by one the dimensions available for choosing a pair of quadrics $(Q_{\text{left}}, Q_{\text{right}})$.

We may detect if the admissible cover $\alpha : X \to \mathbf{P}_a$ in $\Sigma_{d,a}$ has balanced F-bundle only using properties of the maximally connected $f(\alpha) \in MC_{d-2,a}$.

Lemma 4.14. *A cover $\alpha : X \to \mathbf{P}_a$ in $\Sigma_{d,a}$ has balanced F-bundle if and only if $f(\alpha) \in MC_{d-2,a}$ is quadric-generic and has transverse residues.*

Proposition 4.4 – and hence theorem 4.1 – follows from lemmas 4.5, 4.9 and 4.14 and the following claims, which will be proved in section 5.

Claim 4.15. *A general maximally connected chain is quadric-generic.*

Claim 4.16. *A general maximally connected chain has transverse residues.*

Proof of lemma 4.14. We will first translate the F-bundle of $\alpha : X \to \mathbf{P}_a$ into the language of quadrics in $\mathbf{P}^{d-2}$, and then use theorem 3.9 to translate the balancedness of F_α into the statement that $f(\alpha)$ is quadric-generic and has transverse residues.

Start by recalling the geometric interpretation of the directrix of the F-bundle for genus zero covers.

Lemma 4.17. *There is a map of vector bundles over $\mathbf{P}_a$:*

$$\Phi\colon F \to \mathbf{P}_a \times H^0(\mathbf{P}^{d-2}, \mathcal{O}_{\mathbf{P}^{d-2}}(2))$$

such that:

(1) Φ_t sends the vector space F_t to the vector space of quadrics vanishing at the d points $\alpha^{-1}(t) \subset X \subset \mathbf{P}^{d-2}$.

(2) For each component $R_i \subset X$, the directrix $W_i \subset F_i$ maps under Φ_t to the vector space $H^0(I_{R_i}(2))$ of quadrics containing the rational normal curve R_i.

Proof. This is contained in the proof of Lemma 2.7, part (b). □

Our analysis now breaks up into two cases, depending on the parity of the degree d. Recall the language of left and right flags introduced in section 3.1.1.

Lemma 4.18. *Let $\alpha : X \to \mathbf{P}_a \in \Sigma_{d,a}$ be given. Assume that d is even and $f(\alpha) = R_1 \cup \ldots \cup R_a$ is quadric-generic. For any node $p_j \in \mathbf{P}_a$, the left flag $L_j^\bullet = (L_j^1 \subset L_j^2 \subset \ldots \subset L_j^k)$ of F_α may be described as follows:*

(1) The total space: L_j^k is the vector space of quadrics in $\mathbf{P}^{d-2}$ containing the d anchor points over the j-th node.

(2) L_j^{k-l} is the vector space of quadrics in $\mathbf{P}^{d-2}$ containing the sub-chain $R_{j-l+1} \cup \ldots \cup R_j$.

(3) The codimension of $L_j^{k-l} \subset L_j^k$ is $l(d-3)$.

(4) The length k of the flag $L_j^\bullet$ is equal to the minimum of $j+1$ and $\frac{d}{2}$.

Proof. The proof is by induction on j. For $j = 1$, the description above follows directly from the definition of the flag and lemma 4.17. Supposing the description is valid for j, we now prove its validity for $j+1$.

The right hand modification of the flag $L_j^\bullet$ consists of three types of spaces:

- Intersections of $L_j^{k-l} :=$ {quadrics containing $R_{j-l+1}, \dots, R_j$} with the directrix $W_{j+1} :=$ {quadrics containing R_{j+1}}. That is, the quadrics containing the range from R_{j-l+1} up to R_{j+1}. This has the expected dimension since $f(\alpha)$ is quadric-generic.
- The directrix W_{j+1} itself, that is, quadrics containing R_{j+1}.
- The spans $L_j^i + W_{j+1}$. But these will all be equal to the total space of quadrics containing the d anchor points. Indeed, W_{j+1} has codimension $d-3$, and the smallest space L_j^l has dimension at least $d-3$, by induction. If the span $W_{j+1} + L_j^l$ were a proper subspace of $F|_{p_{j+1}}$, then the intersection $L_j^l \cap W_{j+1}$ would have dimension larger than expected, which would violate being quadric-generic.

□

From theorem 3.9 and lemma 4.18 it follows straightforwardly that if d is even and $f(\alpha)$ is quadric-generic then α has a balanced F-bundle. Conversely, if $f(\alpha)$ is not quadric-generic, it is easy to see that the bundle of quadrics of α does not have transverse-flags, and therefore is not balanced. This settles lemma 4.14 for even degrees d.

Here is the analogue of lemma 4.18 for odd degrees.

Lemma 4.19. *Let d be odd and $f(\alpha) = R_1 \cup R_2 \cup \dots \cup R_a$ be quadric-generic and have transverse residues. The elements of the left flag $L_j^\bullet$ are of the following form:*

(1) The total space: Quadrics containing the anchor points $p_1, \dots, p_d$ over the j-th node. This is a vector space of dimension $d(d-3)/2$.
(2) Quadrics containing a subchain of rational curves $R_{j-l+1} \cup \dots \cup R_j$. This has dimension $(d-2l)\frac{d-3}{2}$. Note that $l < \frac{d}{2}$.
(3) Quadrics containing the subchain $R_{j-l+1} \cup \dots \cup R_j$, and whose residual intersection with R_{j-l} is equal to the residual intersection of a quadric containing the length $\frac{d-1}{2}$ subchain $R_{j-l-\frac{d-1}{2}} \cup \dots \cup R_{j-l-1}$. We will call this the restriction condition on R_{j-l}. *This flag element has dimension $(d-2l-1)\frac{d-3}{2}$.*

The proof is analogous to the proof of lemma 4.18. The third case comes from the span construction.

To establish balancedness of the F-bundle, theorem 3.9 tells us we need to establish transverse directrices. This amounts to checking the following three conditions:

- The linear series of quadrics containing an arbitrary subchain $R_i \cup \dots \cup R_j$ has the expected dimension. This follows from $f(\alpha)$ being quadric-generic.
- The linear series of quadrics containing an arbitrary subchain $R_i \cup \dots \cup R_j$ and satisfying the restriction condition on R_{i-1} has the expected

dimension. In many situations, this follows from being quadric-generic , since the restriction map

$$H^0(\mathbf{P}^n, \mathcal{I}_{R_i \cup \ldots \cup R_j}(2)) \to H^0(R_{i-1}, \mathcal{I}_{\text{anchor}}(2))$$

is surjective as long as the length of the range $R_{i-1} \cup \ldots \cup R_j$ is at most equal to $\frac{d-1}{2}$. Indeed, the kernel corresponds to quadrics containing the range from $i-1$ to j, which we know the dimension of by being quadric-generic.

The only extremal case to check is when the subchain $R_i \cup \ldots \cup R_j$ has length $\frac{d-1}{2}$. In this situation, the property we require is precisely having transverse residues (see definition 4.12).

- The linear series of quadrics containing an arbitrary subchain $\bar{X} = R_i \cup \ldots \cup R_j$ and satisfying the restriction condition on both R_{i-1} and R_{j+1} has the expected dimension. Again, the restriction map

$$H^0(\mathbf{P}^n, \mathcal{I}_{R_i \cup \ldots \cup R_j}(2)) \to H^0(R_{i-1}, \mathcal{I}_{\text{anchor}}(2)) \oplus H^0(R_{j+1}, \mathcal{I}_{\text{anchor}}(2))$$

will be surjective for subchains $\bar{X}$ of length less than $\frac{d-3}{2}$, since we can control the dimension of the kernel by being quadric-generic. This implies that the restriction conditions on R_{i-1} and R_{j+1} are independent.

On the other hand, if the length of the subchain $\bar{X}$ is bigger than $\frac{d-3}{2}$, there are no such quadrics by the previous item. Hence, the only case left to consider is subchains $\bar{X}$ of length exactly $\frac{d-3}{2}$.

In this case, let V be the space of quadrics containing $\bar{X}$ and satisfying the restriction condition on R_{i-1}. The space V has the expected dimension $d-3$, by the previous item. Moreover, the restriction map

$$V \to H^0(R_{j+1}, \mathcal{I}_{\text{anchor}}(2))$$

is an isomorphism, since a non-zero kernel would violate having transverse residues. Therefore, the linear series of quadrics containing the subchain $R_i \cup \ldots \cup R_j$ and satisfying the restriction condition on both R_{i-1} and R_{j+1} has the expected dimension, as we wanted to show.

Furthermore, the argument above shows that for odd degrees d, whenever $f(\alpha)$ is quadric-generic and has transverse residues, the F-bundle of α is balanced. Conversely, if one of these conditions fail, we can show that the F-bundle does not have transverse directrices, and therefore is not balanced. This completes the proof of lemma 4.14. □

5. The maximal rank problem for maximally connected chains

The previous section demonstrated that claims 4.15 and 4.16 were exactly what was necessary to conclude theorem 4.1 (and the Main Theorem as a

corollary). In this section, we provide these necessary facts. The key result is the following.

Theorem 5.1. *For a general maximally connected chain* $X \in MC_{r,n}$*, the restriction map*

$$\rho : H^0(\mathbf{P}^r, \mathcal{O}_{\mathbf{P}^r}(2)) \to H^0(X, \mathcal{O}_X(2))$$

has maximal rank.

Note that claim 4.15 follows from theorem 5.1 and the fact that the forgetful map

$$\text{subchain}_{n_1,n_2} : MC_{r,n} \to MC_{r,n_2-n_1+1}$$

sending $R_1 \cup \ldots \cup R_n$ to the subchain $R_{n_1} \cup R_{n_1+1} \cup \ldots \cup R_{n_2}$ is dominant.

We will defer the proof of claim 4.16 to section 5.1, since the argument will seem more natural after studying the proof of theorem 5.1.

We can rephrase theorem 5.1 as a way to determine how many quadrics contain a general maximally connected chain.

Lemma 5.2. *For any maximally connected chain* $X \subset \mathbf{P}^r$ *with* $r \geq 3$*, we have*

$$h^0(\mathcal{I}_X(2)) - h^1(\mathcal{I}_X(2)) = \frac{(r-1)(r+2-2n)}{2}.$$

Proof. Consider the standard short exact sequence

$$0 \to \mathcal{I}_X(2) \to \mathcal{O}_{\mathbf{P}^r}(2) \to \mathcal{O}_X(2) \to 0.$$

Since $H^1(\mathcal{O}_{\mathbf{P}^r}(2)) = 0$, the long exact sequence gives

$$\begin{aligned} h^0(\mathcal{I}_X(2)) - h^1(\mathcal{I}_X(2)) &= h^0(\mathcal{O}_{\mathbf{P}^r}(2)) - h^0(\mathcal{O}_X(2)) \\ &= \binom{r+2}{2} - (2nr - (n-1)(r+1) + 1) + h^1(\mathcal{O}_X(2)) \\ &= \frac{(r-1)(r+2-2n)}{2} + h^1(\mathcal{O}_X(2)). \end{aligned}$$

We are left with showing that $h^1(\mathcal{O}_X(2)) = 0$.

By Serre duality, $h^1(\mathcal{O}_X(2)) = h^0(\omega_X(-2))$. The line bundle $\omega_X(-2)$ has degree $-r$ on R_1 and R_n, and degree 2 on the other components. Let σ be a global section of $\omega_X(-2)$. Since $-r < 0$, it vanishes on R_1, and hence also on $R_1 \cap R_2$. So the section restricted to R_2 has at least $r+2$ zeros, but degree 2. Hence it vanishes identically on R_2 as well, and so on for each component. Therefore, $\sigma = 0$, as we wanted to show. □

Taking lemma 5.2 into account, theorem 5.1 says that

$$h^0(\mathcal{I}_X(2)) = \begin{cases} 0, \text{ if } n \geq \frac{r+2}{2}, \\ \frac{(r-1)(r+2-2n)}{2}, \text{ else.} \end{cases} \tag{14}$$

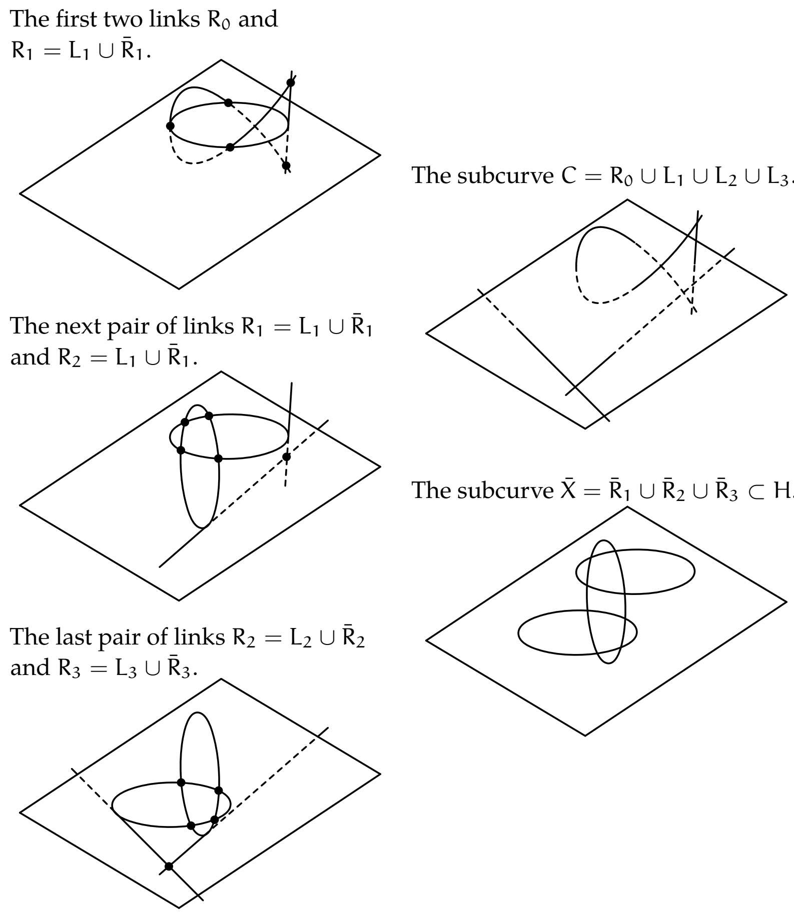


FIGURE 5. The maximally connected chain X used in the induction argument.

Proof of theorem 5.1. We will use induction on the ambient dimension r. The base case $r = 3$ translates into the following three simple facts:

- A twisted cubic is contained in a three dimensional linear system of quadrics.
- There is a unique quadric containing a pair of twisted cubics meeting at five points.
- By picking the third link R_3 generically, the resulting length 3 maximally connected chain X of twisted cubics is not contained in any quadric.

For the induction step, we use the following key construction suggested to us by Eric Larson:

Construction 5.3. $X = C \cup \overline{X} = R_0 \cup R_1 \cup \ldots \cup R_{n-1}$ is a maximally connected chain, constructed as follows.

- Take $R_0 \subset \mathbf{P}^r$ a smooth rational normal curve.
- Choose a general hyperplane $H \subset \mathbf{P}^r$.
- Let L_1 be a general 2-secant line to R_0.
- Let $\overline{R}_1 \subset H$ be a rational normal curve (of degree $r-1$) containing the r points $R_0 \cap H$ and the point $p := L_1 \cap H$. Set $R_1 = L_1 \cup_p \overline{R}_1$.
- Inductively, choose $\overline{R}_{i+1} \subset H$ a general rational normal curve meeting $\overline{R}_i$ in $r+1$ general points. That is, the curve $\overline{X} := \overline{R}_1 \cup \overline{R}_2 \cup \ldots \cup \overline{R}_{n-1}$ is a maximally connected chain of length $n-1$ in $H = \mathbf{P}^{r-1}$.
- Inductively choose L_i to be a general line joining L_{i-1} and $\overline{R}_i$. Set $R_i = L_i \cup \overline{R}_i$.
- Let $C = R_0 \cup L_1 \cup L_2 \cup \ldots \cup L_{n-1}$.

See fig. 5 for a diagrammatic representation of this construction.

We will show that X as above satisfies the maximal rank condition for quadrics. That is, we will show:

- If $n > \frac{r+2}{2}$, then there are no quadrics containing X.
- Otherwise there is a $\frac{(r-1)(r+2-2n)}{2}$ dimensional space of quadrics containing X.

We do this by looking at the sequence

$$0 \to \mathcal{O}_{\mathbf{P}^r}(1) = \mathcal{I}_H(2) \to \mathcal{O}_{\mathbf{P}^r}(2) \to \mathcal{O}_H(2) \to 0$$

and tensoring it with $\mathcal{I}_X$ to get

$$0 \to \mathcal{I}_{C\subset\mathbf{P}^r}(1) \to \mathcal{I}_{X\subset\mathbf{P}^r}(2) \to \mathcal{I}_{\overline{X}\subset H}(2) \to 0. \tag{15}$$

By induction we know that $\overline{X} \subset H$ satisfies the maximal rank condition, and the sequence (15) will let us understand the quadrics containing X.

The case $n > \frac{r+2}{2}$. Then $n-1 \geq \frac{(r-1)+2}{2}$ and by induction and (14), we can assume that $h^0(\mathcal{I}_{\overline{X}\subset H}(2)) = 0$. Hence,

$$h^0(\mathcal{I}_{X\subset\mathbf{P}^r}(2)) = h^0(\mathcal{I}_{C\subset\mathbf{P}^r}(1))$$

and the latter is zero, since C is not contained in any hyperplane (the component R_0 is non-degenerate). Hence, $h^0(\mathcal{I}_{X\subset\mathbf{P}^r}(2)) = 0$, as we wanted to show.

The case $n \leq \frac{r+2}{2}$. We have $n-1 < \frac{(r-1)+2}{2}$, and by induction we can assume

$$h^0(\mathcal{I}_{\overline{X}\subset H}(2)) = \frac{(r-2)(r+3-2n)}{2} = \frac{(r-1)(r+2-2n)}{2} + n - 2.$$

Hence, we want to show that the inclusion

$$H^0(\mathcal{I}_{X\subset\mathbf{P}^r}(2)) \subset H^0(\mathcal{I}_{\overline{X}\subset H}(2)) \tag{16}$$

has codimension $n-2$. The inclusion (16) factors as

$$H^0(\mathcal{I}_{X\subset\mathbf{P}^r}(2)) \subset H^0(\mathcal{I}_{\overline{X}\cup R_0\cup L_1}(2)) \subset H^0(\mathcal{I}_{\overline{X}\subset H}(2)) \tag{17}$$

and the latter inclusion is actually an isomorphism, since it fits in the sequence

$$H^0(\mathcal{I}_{R_0\cup L_1}(1)) \to H^0(\mathcal{I}_{\overline{X}\cup R_0\cup L_1}(2)) \to H^0(\mathcal{I}_{\overline{X}\subset H}(2)) \to H^1(\mathcal{I}_{R_0\cup L_1}(1))$$

where the first and last groups vanish, as deduced from the sequence

$$0 \longrightarrow H^0(\mathcal{I}_{R_0\cup L_1}(1)) \longrightarrow H^0(\mathcal{O}_{\mathbf{P}^r}(1)) \longrightarrow H^0(\mathcal{O}_{R_0\cup L_1}(1)) \longrightarrow H^1(\mathcal{I}_{R_0\cup L_1}(1)) \longrightarrow H^1(\mathcal{O}_{\mathbf{P}^r}(1)) = 0$$

and the fact that the map $H^0(\mathcal{O}_{\mathbf{P}^r}(1)) \to H^0(\mathcal{O}_{R_0\cup L_1}(1))$ is an isomorphism (in other words, $R_0 \cup L_1$ is linearly normal, i.e., it is embedded with the full linear series).

So we have translated our problem into showing that the first inclusion in (17), that is $H^0(\mathcal{I}_{X\subset\mathbf{P}^r}(2)) \subset H^0(\mathcal{I}_{\overline{X}\cup R_0\cup L_1}(2))$, has codimension $n-2$. But this factors naturally in a sequence of $n-2$ inclusions:

$$\begin{aligned} H^0(\mathcal{I}_X(2)) = H^0(\mathcal{I}_{\overline{X}\cup R_0\cup L_1\cup L_2\cup\ldots L_{n-1}}(2)) &\subset H^0(\mathcal{I}_{\overline{X}\cup R_0\cup L_1\cup L_2\cup\ldots\cup L_{n-2}}(2)) \\ &\subset \ldots \\ &\subset H^0(\mathcal{I}_{\overline{X}\cup R_0\cup L_1\cup L_2}(2)) \\ &\subset H^0(\mathcal{I}_{\overline{X}\cup R_0\cup L_1}(2)) \end{aligned} \tag{18}$$

So it is enough to show the following.

Proposition 5.4. *Each inclusion in* (18) *is strict. In other words, for each* $i = 2, \ldots, n-1$, *there exists a quadric containing* $\overline{X} \cup R_0 \cup L_1 \cup \ldots \cup L_{i-1}$, *which does not contain the line* L_i.

We establish proposition 5.4 below, completing the proof of theorem 5.1. □

The key lemma needed for the proof of proposition 5.4 is the following:

Lemma 5.5. *Let* $L \subset \mathbf{P}^r$ *be a line, and* $H \subset \mathbf{P}^r$ *a hyperplane transverse to* L, *and* $R \subset H$ *a rational normal curve. Let* $V \subset H^0(\mathcal{I}_{L\cup R}(2))$ *be a non-empty linear series. Assume:*

(a) *no element of* V *is a reducible quadric containing* H,
(b) *the general element* $Q_{gen} \subset \mathbf{P}^r$ *of* V *is such that* $L \not\subset \operatorname{Sing} Q_{gen}$.

Then for general choices of $p \in L$ *and* $q \in R$*:*

- *The line* $L_{gen} = \overline{pq}$ *is not contained in* Q_{gen}.
- *Among the quadrics* $Q \in V$ *containing* L_{gen}, *a general one does not contain* L_{gen} *in its singular locus.*

Proof of proposition 5.4 assuming lemma 5.5. We want to show that each inclusion in (18) is strict. Let us start with the last one. We set $V = H^0(\mathcal{I}_{\overline{X} \cup R_0 \cup L_1}(2))$, $L = L_1$, and $R = \overline{R}_2$. None of the elements of V are reducible quadrics, because they contain the non-degenerate curve R_0. Thus, condition (a) is met.

Let us check that the general quadric in V is not singular all along L_1. It is enough to show that the restriction of the quadric to H is smooth at $L_1 \cap H$. The restriction of the linear system $H^0(\mathcal{I}_{\overline{X} \cup R_0 \cup L_1 \subset \mathbf{P}^r}(2))$ to H is $H^0(\mathcal{I}_{\overline{X} \subset H}(2))$, which only depends on $\overline{X}$. The general quadric in H containing $\overline{X}$ cannot be singular all along $\overline{R}_1$, because $\overline{R}_1$ is a non-degenerate curve in H, and the singular locus of a quadric is always a linear space. Hence, as long as we pick $L_1 \cap H$ to be a general point in $\overline{R}_1$, we will be fine. We can do this, as a matter of fact in construction 5.3 we may first choose $\overline{X}$ and any $r + 1$ distinct points in $\overline{R}_1$, and then we can find an $L_1 \cup R_0 \subset \mathbf{P}^r$ meeting H exactly at these points.

The hypotheses of lemma 5.5 are satisfied, so its conclusion says that for a general choice of L_2, the inclusion (18) will be strict.

Moreover, lemma 5.5 also says that a general quadric in $H^0(\mathcal{I}_{\overline{X} \cup R_0 \cup L_1 \cup L_2}(2))$ is not singular all along L_2. Thus we conclude proposition 5.4 by iteratively applying lemma 5.5.

□

Hence, we will be done with the proof of theorem 5.1 as soon as we establish lemma 5.5.

Proof of lemma 5.5. For the first part, assume that $L_{gen} \subset Q_{gen}$. Then Q_{gen} contains the join $J(L, R)$. But the tangent space to $J(L, R)$ at $p \in L$ is r-dimensional, because the cone $C_p R$ with vertex p over R is contained in $J(L, R)$. Hence, Q_{gen} is singular along L as well, but we assumed this was not the case – contradiction.

Now let us show that among the quadrics in V containing $L_{gen} = \overline{pq}$, a general one does not contain L_{gen} in its singular locus. Pick $p \in L$ such that the general element of V is not singular at p (this can be done by our assumption). Now there are two cases.

Case 1: One possibility is that as we vary $q \in R$, we get a non-trivial family of hyperplanes consisting of quadrics in V containing $\overline{pq}$. But then the union of these hyperplanes is the whole vector space V, and we know that the general quadric in V is smooth at p. Hence, it is smooth at the general point of the line $\overline{pq}$ as well.

Case 2: Otherwise, every quadric in V containing $\overline{pq}$ also contains the cone $C_p R$. Pick any of these quadrics, Q. Then Q cannot be singular all along R, because the singular locus of Q is a linear space, and R spans the hyperplane H. Hence, for some $q \in R$, the quadric Q is smooth

at q. Take the line $\overline{pq}$ which is contained in C_pR, and hence in Q. As Q is smooth at q, it is smooth at the general point of $\overline{pq}$.

In any case, we still conclude lemma 5.5. □

5.1. **Transverse residues.** We now adapt construction 5.3 to prove claim 4.16.

Proof of claim 4.16. From now on the ambient dimension r will be odd. We want to show that a general maximally connected chain $X \subset \mathbf{P}^r$ of length $r+2$ has transverse residues, as in definition 4.12. This means we partition X in three groups: the middle link R_{middle}, and the $\frac{r+1}{2}$ links coming before (X_{left}) and after (X_{right}) it. We want to show that a pair of quadrics Q_{left} and Q_{right} containing X_{left} and X_{right} respectively cannot have the same residual intersections with R_{middle}.

It is enough to exhibit a single example of such an X. Consider an X with the following properties:

- R_{middle} is a smooth rational normal curve
- $R_{\text{middle}} \cup X_{\text{right}}$ and $X_{\text{left}} \cup R_{\text{middle}}$ are exactly as in construction 5.3. That is, to construct X_{right}, choose a hyperplane H and let $R_i = L_i \cup \overline{R}_i$, where:
  - L_1 is a general secant line to R_{middle}
  - $\overline{R}_1$ is a general rational normal curve in the hyperplane H containing $R_{\text{middle}} \cap H$ and $L_1 \cap H$
  - $\overline{R}_1 \cup \overline{R}_2 \cup \overline{R}_3 \cup \ldots \cup \overline{R}_{\frac{r+1}{2}} \subset H$ form a general maximally connected chain in H.
  - L_i is a general line joining L_{i-1} and $\overline{R}_i$ for $i = 2, \ldots, \frac{r+1}{2}$.

We claim that such a curve X has transverse residues. Let Λ_{right} be the linear span of $L_1, L_2, \ldots, L_{\frac{r+1}{2}}$. Any quadric containing X_{right} must split as $H \cup H_{\text{right}}$, where H_{right} is a hyperplane containing Λ_{right}. Otherwise, the restriction of such quadric to H would contain a general maximally connected chain of length $\frac{r+1}{2}$, which theorem 5.1 says cannot exist.

The $r-2$ points of residual intersection of the reducible quadric $H \cup H_{\text{right}}$ with R_{middle} is the set $R_{\text{middle}} \cap H_{\text{right}}$ minus the pair of points in $L_1 \cap R_{\text{middle}} = \Lambda_{\text{right}} \cap R_{\text{middle}}$. Therefore, to see how the residual points vary in R_{middle}, we only need to remember the linear space Λ_{right}.

Define Λ_{left} analogously. Note that we may choose the pair of $(\Lambda_{\text{right}}, \Lambda_{\text{left}})$ arbitrarily, as long as they meet R_{middle} in a pair of points each. This is because when performing construction 5.3, we may first choose the lines L_i arbitrarily, and then choose fitting $\overline{R}_i$.

So we want to show that for Λ_{right} and Λ_{left} generic, there are no hyperplanes H_{left} and H_{right} containing the respective linear spaces $\Lambda_{\text{left}}, \Lambda_{\text{right}}$

whilst having the same residual intersection on R_{middle}. Consider the incidence correspondence:

$$\begin{aligned}\Sigma := \big\{(\Lambda_{\text{left}}, \Lambda_{\text{right}}, H_{\text{left}}, H_{\text{right}}, p_1, \ldots, p_{r-2}) \mid \\ &\#(\Lambda_{\text{left}} \cap R_{\text{middle}}) = \#(\Lambda_{\text{right}} \cap R_{\text{middle}}) = 2, \\ &\Lambda_{\text{left}} \subset H_{\text{left}},\ \Lambda_{\text{right}} \subset H_{\text{right}}, \\ &\{p_1, \ldots, p_{r-2}\} \cup (\Lambda_{\text{left}} \cap R_{\text{middle}}) = H_{\text{left}} \cap R_{\text{middle}}, \\ &\{p_1, \ldots, p_{r-2}\} \cup (\Lambda_{\text{right}} \cap R_{\text{middle}}) = H_{\text{right}} \cap R_{\text{middle}}\big\}\end{aligned}$$

That is, the points p_i are the (common) residual intersection of the hyperplanes $\Lambda \subset H$ with R_{middle}. The incidence correspondence Σ admits a projection map to:

$$\Sigma' = \big\{(\Lambda_{\text{left}}, \Lambda_{\text{right}}) \mid \#(\Lambda_{\text{left}} \cap R_{\text{middle}}) = \#(\Lambda_{\text{right}} \cap R_{\text{middle}}) = 2\big\}$$

We want to show that the map $\Sigma \to \Sigma'$ is not dominant. We do this by showing that $\dim \Sigma < \dim \Sigma'$, as follows.

$$\begin{aligned}\dim \Sigma &= r - 2 && \text{(choose the } p_i\text{'s)} \\ &\quad + 2 + 2 && \text{(choose } \Lambda_{\text{left}} \cap R_{\text{middle}}, \Lambda_{\text{right}} \cap R_{\text{middle}}) \\ &\quad + 0 + 0 && (H_{\text{left}}, H_{\text{right}} \text{ are the span of } p_i\text{'s and } \Lambda \cap R_{\text{middle}}) \\ &\quad + \frac{(r-3)(r-1)}{4} && \text{(choose } \Lambda_{\text{right}} \subset H_{\text{right}} \text{ containing } \Lambda_{\text{right}} \cap R_{\text{middle}}) \\ &\quad + \frac{(r-3)(r-1)}{4} && \text{(same for } \Lambda_{\text{left}}) \\ &= \frac{(r-1)^2}{2} + 3\end{aligned}$$

while

$$\begin{aligned}\dim \Sigma' &= 2 + 2 && \text{(choose pair points } \Lambda \cap R_{\text{middle}}) \\ &\quad + \frac{(r-1)^2}{4} && \text{(choose } \Lambda_{\text{right}} \text{ containing } \Lambda_{\text{right}} \cap R_{\text{middle}}) \\ &\quad + \frac{(r-1)^2}{4} && \text{(same for } \Lambda_{\text{left}}) \\ &= \frac{(r-1)^2}{2} + 4 \\ &= \dim \Sigma + 1\end{aligned}$$

which is as we desired. This completes the proof of claim 4.16. □

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